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On the admissibility of repeated irreducible representations in barnacle-free unique-spin unique-mass relativistic wave equations

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Abstract. Considering relativistic wave equations in the first-order form, wherein the transformation of the wavefunction under the proper Lorentz group involves a certain number of inequivalent irreducible representations (IIRs) repeated an arbitrary number of times, we note some of the restrictions (on the skeleton matrix of the matrices β^{μ} occurring in the equation and on the spin blocks of β^0) which arise from the requirements (i) that the equation be barnacle free and (ii) that there be solutions corresponding to a single spin and single mass only (without any degeneracy). We show that if the number of IIRs is just two, these restrictions permit only two types of equations with no repeated IRs in either case. We also consider equations involving three IIRs with arbitrary multiplicity, carry out a reduction of the skeleton matrix, and analyse the implications of the above mentioned requirements with regard to the possible existence of equations in which the multiplicity of one of the IRs is the sum of the multiplicities of the other two. Nothing is assumed about the specific IRs involved, except that they are linked.

1. Introduction

The problem of the consistency of relativistic wave equations for spinning particles in the presence of external fields acting on them, has been studied extensively during the past two decades (Johnson and Sudarshan 1961, Velo and Zwanziger 1969a,b, Velo and Wightman 1978, Wightman 1968, 1971, Federbush 1961, Hagen 1972, Mathews 1974). It is now well recognised that the various familiar equations for higher spin such as the Rarita–Schwinger equation (1941) for spin-$\frac{3}{2}$ and the Fierz–Pauli equation for spin-2 (1939) became subject to several types of pathologies when an interaction with other (external) fields is introduced. For this reason (among others) the construction and investigation of new types of wave equations have been taken up by several workers (Glass 1971, Capri 1969, 1972, Khalil 1977, Hurley and Sudarshan 1975, Fisk and Tait 1973). The new spin-$\frac{3}{2}$ equation due to Glass (1971), equations for spin-$\frac{3}{2}$ derived by Capri (1969, 1972) and Khalil's spin-$\frac{1}{2}$ equation (1977) are examples of such equations. They are all of the general form

$$(-\mathrm{i}\beta^{\mu}\partial_{\mu}+m)\psi=0. \tag{1.1}$$

The new element in the derivation of these equations is the deliberate introduction of a non-trivial multiplicity in the occurrence of certain irreducible representations (IRs) in the representation $T(\Lambda)$ of the proper Lorentz group $\mathscr{L}_{+}^{\uparrow}$ according to which the wavefunction is required to transform.

$$T(\Lambda)=\sum \alpha_{\tau} D^{\tau}(\Lambda) \tag{1.2}$$

where α_τ is the multiplicity of the IR D^τ (we shall hereafter simply write τ for the IR; τ itself is a short notation for (m, n) where m, n are non-negative integers or half-integers). Once the representation $T(\Lambda)$ is specified, the matrices β^μ in (1.1) get determined (according to the theory of Bhabha (1945, 1949), Wild (1947), Gel'fand and Yaglom (1948a,b) to within a certain number of arbitrary parameters forming the elements of the skeleton matrix (Mathews *et al* 1980) associated with the representation. In order to ensure that the equation describes particles of unique spin and mass, the elements of the skeleton matrix have to be suitably restricted. However, on carrying this process through in particular cases, one does not necessarily get an equation which is different from some simpler equation involving fewer IRs. Hurley and Sudarshan (1975) (see also Khalil 1976) have pointed out that part of the wavefunction may just be a 'barnacle' whose behaviour is completely determined by the complementary part of the wavefunction, which by itself satisfies a simpler wave equation. In fact, several of the recently proposed new equations (Capri 1969, 1972, Khalil 1977, Hurley and Sudarshan 1975, Fisk and Tait 1973) have been shown to be 'barnacled' and have no more content (though looking more complicated) than known simpler equations. The question then arises: in trying to construct new equations of the form (1.1), is it possible to ensure in advance that the equation will not turn out to be barnacled? Or better still, can one construct exhaustive classes of unbarnacled equations? This paper is devoted to an investigation of these questions. Kahlil (1978) has identified certain conditions on the skeleton matrix β^μ, which are necessary and sufficient for barnacles to be absent. With these conditions as input, we analyse the spin blocks (of which β^0 is a direct sum), and impose on them the conditions for uniqueness and non-degeneracy of mass and spin of particles described by the wave equation. After deducing a few results applicable to the general situation where the number and multiplicity of IRs entering into the Lorentz group representation $T(\Lambda)$ (according to which ψ transforms) are both arbitrary, we show that as far as equations involving two inequivalent IRs are concerned, the general results imply that only two special classes of equations are permitted, with no non-trivial multiplicity in either. We also consider an instructive example of equations involving three inequivalent IRs (with a certain relation between the multiplicities of the IRs). An exhaustive analysis of all such equations will be presented separately.

We wish to emphasise that in contrast to the traditional approach (see for instance Bhabha 1945, 1949) to relativistic wave equations of the form (1.1) which involves the prescription of certain algebraic properties of the matrices β^μ and subsequent investigations of the implications of the algebra in respect of the mass and spin content of the wave equation, we begin by enforcing the requirements of uniqueness and non-degeneracy of mass and spin and the Khalil conditions for the equation not to be the same as some simpler equation camouflaged by superfluous 'barnacles'. While the old approach of Duffin (1938) and Kemmer (1939), Madhava Rao (1942) and Bhabha (1945, 1949) etc, gets too complicated to be carried much further and, in any case, is only an indirect way of placing constraints on the elements of the matrices β^μ, our treatment focuses directly on the skeleton matrix which is the repository of all the freedom remaining after the compulsions of Lorentz invariance are taken into account. We have thus been able to deal with whole classes of equations irrespective of the value of the desired physical spin and without *a priori* specification of multiplicities.

In § 2 we recall the relevant general features of the skeleton matrix of the β^{μ} in barnacle-free theories and also of the spin block $\beta^0_{(j)}$ into which β^0 can be resolved. The unique-mass, unique-spin conditions are expressed in terms of the spin blocks and some of their implications for the Jordan canonical structures of two square matrices X_j, Y_j of which $(\beta^0_{(j)})^2$ is the direct sum, are noted. In § 3, we consider the class of all equations involving two inequivalent IRs with arbitrary multiplicity and show that, for given mass and spin there are just two unbarnacled irreducible equations in this class, namely the Hurley equation and another equation involving the IRs $(s+\frac{1}{2}, \frac{1}{2})$ and $(s, 0)$ once each. Section 4 deals with equations involving three inequivalent IRs. We establish some general properties of spin blocks arising in this case and proceed to use them to study the admissibility of equations in which the multiplicities $\alpha_1, \alpha_2, \alpha_3$, with which the IRs enter, are related by $\alpha_1 = \alpha_2 + \alpha_3$. This is intended to be only an illustrative example. A comprehensive analysis of equations involving three inequivalent IRs with multiplicities left arbitrary will be presented in a separate paper.

2. General considerations

We shall take the wavefunction ψ, as usual, to be a direct sum of parts $\psi^{(\tau,\sigma)}$ transforming according to the IRs τ of the proper Lorentz group $\mathscr{L}^{\uparrow}_{+}$. (The index σ is needed when any IR occurs with a multiplicity, to label the various identical IRs.) We recall (Mathews *et al* 1980a) that the matrix elements of the β^{μ} between the states of any two given IRs τ' and τ decompose into a Lorentz group Clebsch–Gordan coefficient and a reduced matrix element and that these reduced matrix elements (which can be assigned arbitrary values) constitute the skeleton matrix which has a crucial role in determinining the algebraic properties of the β^{μ}. The skeleton matrix C is made up of blocks $C^{(\tau',\tau)}$ associated with particular pairs of IRs; and the block $C^{(\tau',\tau)}$ has $\alpha_{\tau'}$ rows and α_{τ} columns where α_{τ}, typically, is the multiplicity of the IR τ in $T(\Lambda)$.

2.1. *Barnacle-free equations*

It has been shown by Khalil that, in order that no barnacles be present, it is necessary that the following conditions on the ranks of various submatrices of the skeleton matrix be satisfied.

Let $C^{(\tau',\cdot)}$ be the matrix made up of the row of blocks $C^{(\tau',\tau)}$ with τ' fixed and τ varying over all IRs present in $T(\Lambda)$. Its rank must be equal to $\alpha_{\tau'}$, the multiplicity of the IR τ'. Similarly, if $C^{(\cdot,\tau)}$ is the matrix made up of the column of blocks $C^{(\tau',\tau)}$ for given τ, its rank must be equal to α_{τ}. There will be no barnacles if and only if these two conditions are satisfied for all τ and τ'.

In the present work, these conditions will be imposed at the outset.

2.2. *The spin blocks*

If a basis which diagonalises $\boldsymbol{J}^2$ and J_z is employed, it becomes possible to write β^0 in block-diagonal form, as a direct sum of blocks $\beta^0_{(j)}$ associated with the various spins j

occurring in one or more of the IRs (m, n) entering the problem. The spin j block has the form

$$\beta^{0}_{(j)} \times I_{2j+1} \tag{2.1}$$

where $\beta^{0}_{(j)}$ is made up of blocks

$$C^{(\tau',\tau)} g_j^{(\tau',\tau)} \tag{2.2}$$

associated with pairs of IRs τ, τ' *both* of which contain j (i.e. such that $m+n \geqslant j \geqslant |m-n|$ and $m'+n' \geqslant j \geqslant |m'-n'|$). $C^{(\tau',\tau)}$ is itself the (τ', τ) block of the skeleton matrix and $g_j^{(\tau',\tau)}$ is a Lorentz group Clebsch–Gordan coefficient. Each element of $\beta^{0}_{(j)}$ multiplies a unit matrix of dimension $(2j+1)$ as indicated by the direct product in (2.1).

A useful illustration is provided by the example of a theory involving just three irreducible representations

$$\tau_1 = (m, n), \qquad \tau_2 = (m - \tfrac{1}{2}, n + \tfrac{1}{2}), \qquad \tau_3 = (m + \tfrac{1}{2}, n - \tfrac{1}{2}),$$

which is of considerable interest. Taking $m > n$, we see readily that the spins $(m+n)$, $(m+n-1), \ldots, (m-n+1)$ occur in all the three IRs. Therefore, with the skeleton matrix C written as in Mathews *et al* (1980), one has for $(m+n) \geqslant j \geqslant (m-n+1)$,

$$\beta^{0}_{(j)} = \begin{pmatrix} 0 & Ag_j & Bg'_j \\ Dg_j & 0 & 0 \\ Eg'_j & 0 & 0 \end{pmatrix}. \tag{2.3}$$

(Here g_j and g'_j are abbreviations for $g_j^{(\tau_1,\tau_2)}$ and $g_j^{(\tau_1,\tau_3)}$ respectively.) The value $(m-n)$ of the spin occurs in τ_1 and τ_2 but not in τ_3 if $m > n$ and then

$$\beta^{0}_{(m-n)} = \begin{pmatrix} 0 & Ag_{(m-n)} \\ Dg_{(m-n)} & 0 \end{pmatrix}; \tag{2.4}$$

but it occurs only in τ_1 if $m = n$, and then

$$\beta^{0}_{(0)} = 0. \tag{2.5}$$

Finally, j can take one more value $(m-n-1)$ if $m \geqslant n+1$, and it occurs then in τ_2 so that

$$\beta^{0}_{(m-n-1)} = 0. \tag{2.6}$$

2.3. *Uniqueness of mass and spin: non-degeneracy*

Conditions requiring that equation (1.1) should describe particles of unique mass and spin find their most direct expression in terms of the spin blocks. If the equation is to have solutions for only a single spin s, it is necessary that all the $\beta^{0}_{(j)}$ be nilpotent except $\beta^{0}_{(s)}$. The condition for uniqueness of mass is that all non-zero eigenvalues of the various $\beta^{0}_{(j)}$ should be of equal magnitude; they can be required, without loss of generality, to be ± 1. It follows then that, if both the mass and the spin are required to be unique, the only spin block which is allowed to be not nilpotent, $\beta^{0}_{(s)}$, must have its non-zero eigenvalues equal to ± 1. Finally, if there is to be no degeneracy (i.e. there is only one particle of mass m and spin s, together with its antiparticle) each of the eigenvalues ± 1 should appear just once.

2.4. Bipartite form†—the X_j and Y_j matrices

Now, as was shown in the works of Cox (1974a,b) and Mathews *et al* (1980), β^0 can always be written (by suitable ordering of the IRs) in block form consisting of two vanishing diagonal blocks and two (non-vanishing) off-diagonal blocks. Correspondingly, $\beta^0_{(j)}$ can be written as

$$\beta^0_{(j)} = \begin{pmatrix} 0 & U_j \\ V_j & 0 \end{pmatrix} \tag{2.7}$$

with U_j, V_j, themselves made up of blocks $C^{(\tau',\tau)}g_j^{(\tau',\tau)}$. The blocks U_j, V_j are, in general, rectangular, but in

$$(\beta^0_{(j)})^2 \equiv \begin{pmatrix} X_j & 0 \\ 0 & Y_j \end{pmatrix} \tag{2.8}$$

$$X_j = U_j V_j \qquad \text{and} \qquad Y_j = V_j U_j \tag{2.9}$$

are both square matrices. The conditions, stated in the last paragraph, for the wave equation to describe a unique (non-degenerate) particle of specified mass m and spin s lead to the requirement that X_j and Y_j must be nilpotent for every $j \neq s$, while for $j = s$, each of them should have a Jordan canonical form consisting of the direct sum of a single unit eigenvalue and one or more nilpotent irreducible Jordan blocks (NIJBs). The first part of this assertion follows from the fact that for any $j \neq s$, $\beta^0_{(j)}$ has to be nilpotent and hence $(\beta^0_{(j)})^2$ does also. On the other hand, $(\beta^0_{(s)})^2$ has to have one pair of unit eigenvalues, and the second part of our assertion is that these are distributed between X_s and Y_s. Since $\mathrm{Tr}\, X_s = \mathrm{Tr}(U_s V_s) = \mathrm{Tr}(V_s U_s) = \mathrm{Tr}\, Y_s$ and $\mathrm{Tr}(\beta^0_{(s)})^2 = 2 = \mathrm{Tr}\, X_s + \mathrm{Tr}\, Y_s$ we have

$$\mathrm{Tr}\, X_s = \mathrm{Tr}\, Y_s = 1$$

which shows that of the only two non-zero eigenvalues (equal to unity) one is in X_s and the other in Y_s as asserted.

We now proceed to use the above result to determine exhaustively the possibilities for equations involving just two inequivalent IRs (which occur *a priori* with arbitrary multiplicities), and to investigate a special class of three-IIR equations.

3. Two-IIR equations

When only two inequivalent IRs

$$\tau_1 \equiv (m_1, n_1) \qquad \text{and} \qquad \tau_2 \equiv (m_2, n_2) \tag{3.1}$$

are present (with multiplicities α_1 and α_2) in $T(\Lambda)$, the skeleton matrix has the form

$$C = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \tag{3.2}$$

where A is an $\alpha_1 \times \alpha_2$ matrix and B an $\alpha_2 \times \alpha_1$ matrix, both with arbitrary elements. For absence of barnacles, it is necessary that A, considered as $C^{(\tau_1,\cdot)}$ must have rank α_1,

† This term is used to describe the form of β^0 (and of $\beta^0_{(j)}$ introduced below, since the form is a consequence of the bipartite nature of the graphs (Cox 1974a,b) characterising the linking of various IRs by the β^μ.

while, B considered as $C^{(\cdot,\tau_2)}$ must have rank α_2. Taken together, these conditions require

$$\alpha_1 = \alpha_2 = \alpha. \tag{3.3}$$

(say). Consequently A has to be a *non-singular* square matrix (rank = dimension = α) and the same is true for B.

The spin block $\beta^0_{(j)}$ associated with any spin value j contained in both τ_1 and τ_2 is now

$$\beta^0_{(j)} = \begin{pmatrix} 0 & g_j A \\ g_j B & 0 \end{pmatrix} \tag{3.4}$$

where

$$g_j \equiv g_j^{(\tau_1,\tau_2)} = g_j^{(\tau_2,\tau_1)}. \tag{3.5}$$

Considering $(\beta^0_{(j)})^2$, one notes that

$$X_j \equiv g_j^2 AB \qquad \text{and} \qquad Y_j = g_j^2 BA \tag{3.6}$$

are non-singular matrices and hence have all eigenvalues non-zero. However, since we demand non-degeneracy, which permits only a single non-zero eigenvalue each for X_j and Y_j, we are forced to the conclusion that $\alpha = 1$ only is allowed.

Any non-trivial multiplicity being thus ruled out, we have A and B as single numbers, say a and b, and then $X_j = Y_j = g_j^2\, ab$. This must be equal to unity for $j = s$, the desired physical spin

$$ab = (g_s)^{-2}. \tag{3.7}$$

If any $j \neq s$ existed which also occurs in both τ_1 and τ_2 (i.e. for which $g_j \neq 0$) then, for such j, X_j and Y_j would also be non-zero, meaning that non-trivial solutions of the wave equations would exist for spin $j \neq s$ also. Hence, for uniqueness of spin we require that s be the only spin present in both the IRs. The only possibilities consistent with this requirement and with the condition $|m_1 - m_2| = |n_1 - n_2| = \frac{1}{2}$ which is necessary for τ_1 and τ_2 in (3.1) to be linked, are

$$T(\Lambda) \sim (s, 0) \oplus (s - \tfrac{1}{2}, \tfrac{1}{2}) \tag{3.8}$$

which corresponds to the Hurley equation, and

$$T(\Lambda) \sim (s + \tfrac{1}{2}, \tfrac{1}{2}) \oplus (s, 0) \tag{3.9}$$

which has not apparently been considered in the literature for general spin. The Kemmer equation for spin-0 belongs to the class (3.9) with $s = 0$, and the Dirac equation to (3.8) with $s = \frac{1}{2}$. These two special cases are the only ones to admit parity invariance.

4. Three-IIR equations

4.1. *The skeleton matrix*

We consider now equations involving just three distinct IRs τ_1, τ_2, τ_3 occurring with arbitrary multiplicities α_1, α_2, α_3. Let τ_2 and τ_3 be such that they can be linked directly to τ_1 through the β^{μ}, say,

$$\begin{aligned}\tau_1 &= (m, n)\\ \tau_2 &= (m+\tfrac{1}{2}\varepsilon_2, n+\tfrac{1}{2}\eta_2)\\ \tau_3 &= (m+\tfrac{1}{2}\varepsilon_3, n+\tfrac{1}{2}\eta_3)\end{aligned} \tag{4.1}$$

with ε_2, η_2, ε_3 and η_3 equal to $+1$ or -1 independently. Then it is self-evident that τ_2 and τ_3 cannot be directly linked by the β^{μ}. So the skeleton matrix takes the form

$$C = \begin{array}{c} \begin{pmatrix} 0 & A & B \\ D & 0 & 0 \\ E & 0 & 0 \end{pmatrix} \begin{array}{c} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{array} \\ \begin{array}{ccc} \alpha_1 & \alpha_2 & \alpha_3 \end{array} \end{array} \tag{4.2}$$

where A is a short notation for the block $C^{(\tau_1,\tau_2)}$ which has α_1 rows and α_2 columns, and similarly for the other blocks B, D, E.

The conditions of § 2.1 for the absence of barnacles require in the present case that the ranks r of the various submatrices of C be as follows:

$$\begin{aligned} r(A) &= r(D) = \alpha_2\\ r(B) &= r(E) = \alpha_3\\ r(A, B) &= r\begin{pmatrix} D\\ E\end{pmatrix} = \alpha_1.\end{aligned} \tag{4.3a}$$

From these it may be noted that

$$\alpha_1 \geqslant \alpha_2, \alpha_3 \qquad \text{and} \qquad \alpha_2+\alpha_3 \geqslant \alpha_1. \tag{4.3b}$$

Subject to the conditions (4.3a) and (4.3b), the elements of the submatrices A, B, D, E are quite arbitrary, but some of this arbitrariness is not essential.

Similarity transformations which mix equivalent IRs among themselves do not affect the transformation property of the wavefunction, and one can exploit this freedom (see Appendix) to replace the skelton matrix (4.2) by an equivalent one in which

$$D = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \end{pmatrix} \begin{array}{l} (\alpha_1-\alpha_3)\text{ rows} \\ (\alpha_2+\alpha_3-\alpha_1)\text{ rows} \end{array} \tag{4.4a}$$

$$\begin{array}{c} E = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{array}{l} (\alpha_2+\alpha_3-\alpha_1)\text{ rows} \\ (\alpha_1-\alpha_2)\text{ rows.} \end{array} \\ \begin{array}{ccc} (\alpha_1-\alpha_3) & (\alpha_2+\alpha_3-\alpha_1) & (\alpha_1-\alpha_2) \\ \text{columns} & \text{columns} & \text{columns} \end{array} \end{array} \tag{4.4b}$$

We shall refer to these as the *standard* forms of D and E, wherein the unit matrices I and null blocks 0 have dimensions as indicated. In making similarity transformations, the submatrices A and B also get transformed of course. Further similarity transformations are possible which take A and B also to simpler forms without disturbing the above structure of D and E, but the variety of possibilities is such that it is not profitable to consider them at this stage. We shall deal with them in connection with specific classes of theories in a future publication (Mathews and Vijayalakshmi 1980).

4.2. Spin blocks

We observe at the outset that, if the spin values contained in τ_1, τ_2 and τ_3 are examined, most of them would be appearing in all three IRS. We shall refer to these as *common spins*. Any spin value which occurs in just two of the IRS (τ_1 and one other, which will be, by convection, taken as τ_2), will be referred to as a *special spin*. There may also be spin values which occur in only one of the three IRS, but these do not have any significant role, and there will be no further reference to them.

The spin block associated with spin j, namely

$$\beta^0_{(j)} = \begin{pmatrix} 0 & U_j \\ V_j & 0 \end{pmatrix} \tag{4.5}$$

has

$$U_j = Ag_j; \qquad V_j = Dg_j \tag{4.6}$$

if j is a special spin (occurring in τ_1 and τ_2 only), while

$$U_j = (Ag_j, Bg'_j); \qquad V_j = \begin{pmatrix} Dg_j \\ Eg'_j \end{pmatrix} \tag{4.7}$$

if j is a common spin. In either case

$$\begin{aligned} \text{rank of } \beta^0_{(j)} &= \text{rank of } U_j + \text{rank of } V_j \\ &= 2 \times \text{rank of } U_j. \end{aligned} \tag{4.8}$$

This follows from the self-evident fact that the rows of $(0, U_j)$ in (4.5) are linearly independent of those of $(V_j, 0)$.

Consider now the square matrices

$$X_j \equiv U_j V_j \qquad \text{and} \qquad Y_j \equiv V_j U_j. \tag{4.9}$$

As already noted, neither is permitted to have any non-zero eigenvalue if $j \neq s$, while each has one unit eigenvalue if $j = s$. So it follows that, whenever the dimension of either of these matrices exceeds unity, its Jordan canonical form must necessarily include one or more NIJBS. Conversely, if X_j or Y_j is non-singular, its dimension must necessarily be unity.

We shall now illustrate the power of the above results in the determination of admissible types of equations, by considering a special class.

4.3. The 'stretched case; $\alpha_1 = \alpha_2 + \alpha_3$

In this case α_1 is stretched to the highest value it can take in barnacle-free theories for given α_2 and α_3 of equation (4.3*b*). In the skeleton matrix C, the number of rows α_1 of the submatrix $(A\ B)$ then becomes equal to the number of columns, $\alpha_2 + \alpha_3$. Further, for absence of barnacles, the rank also is to be α_1. Thus $(A\ B)$ constitutes a non-singular square matrix, and so does $\begin{smallmatrix} D \\ E \end{smallmatrix}$, for similar reasons.

Consider now the spin block $\beta^0_{(j)}$ for a common spin j

$$\beta^0_{(j)} = \begin{pmatrix} 0 & g_j A & g'_j B \\ g_j D & 0 & 0 \\ g'_j E & 0 & 0 \end{pmatrix}. \tag{4.10}$$

It is evident that $U_j \equiv (g_j A,\ g'_j B)$ has the same rank α_1 as $(A\ B)$; so also for V_j.

Consequently, in $(\beta^0_{(j)})^2$, $X_j \equiv U_jV_j$ and $Y_j \equiv V_jU_j$, are both non-singular matrices. But we permit non-zero eigenvalues for X_j and Y_j only for $j = s$, the particular spin which it is desired the particle should have. Therefore, if we are to have a unique-spin theory, there must be only a single $j = s$ for which the spin block is of the above form. (This would restrict n in (4.1) to be $\frac{1}{2}$ or 1 depending on the values of the ε's and η's).

While uniqueness of spin can thus be enforced, it turns out that uniqueness and non-degeneracy of mass cannot. The problem is that the matrices X_s and Y_s which are non-singular are of dimension $\alpha_1 \equiv \alpha_2 + \alpha_3 \geqslant 2$ (since $\alpha_2, \alpha_3 \geqslant 1$) and therefore violate the condition deduced in § 4.2 that X_s and Y_s, if non-singular, must be of unit dimension.

Therefore, there are no admissible equations in the class characterised by $\alpha_1 = \alpha_2 + \alpha_3$.

5. Summary

In this paper we have developed a technique to construct and analyse relativistic wave equations when repeated IRs are allowed with arbitrary (unspecified) multiplicities and transformation properties. Making use of this technique we study equations involving two IIRs as well as a particular class of equations involving three IRs. In spite of the generality of the initial input, we find that only two cases survive, namely, Hurley's equations based on the representation $(s, 0) \oplus (s - \frac{1}{2}, \frac{1}{2})$ and the rather similar equations involving the representation $(s, 0) \oplus (s + \frac{1}{2}, \frac{1}{2})$ with unit multiplicity. Of these, the Dirac equation for spin-$\frac{1}{2}$ and the Duffin–Kemmer–Petiau equation for spin-0 are the only ones which have parity invariance. Also, in the case of three IIRs with the multiplicities arranged as $\alpha_1 = \alpha_2 + \alpha_3$, no unique-spin, unique-mass equation can be accommodated. A consideration of more general cases of equations involving three IIRs requires the use of certain properties of matrices (especially nilpotent matrices) which one hardly ever finds applied. A presentation of these together with an exhaustive analysis of equations with three IIRs will be given in a separate paper (Mathews and Vijayalakshmi 1980).

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Appendix

The wavefunction is assumed to be taken in the Lorentz reduced form

$$\psi = \begin{bmatrix} \psi^{(1)} \\ \psi^{(2)} \\ \vdots \\ \cdot \end{bmatrix} \qquad \text{with} \qquad \psi^{(i)} = \begin{bmatrix} \psi^{(i,1)} \\ \psi^{(i,2)} \\ \vdots \\ \psi^{(i,\alpha_i)} \end{bmatrix} \tag{A.1}$$

wherein, under a Lorentz transformation Λ,

$$\psi^{(i,\sigma)} \to D^{\tau_i}(\Lambda)\psi^{(i,\sigma)}; \qquad \sigma = 1, \ldots, \alpha_i. \tag{A.2}$$

Since any linear combination of the $\psi^{(i,\sigma)}$ for fixed i will also transform according to the IR τ_i, the Lorentz reduced form will be preserved and the transformation law (A.2) will remain unaffected if the following change of basis mixing identical IRs is made:

$$\psi^{(i,\sigma)} \to \sum_{\sigma'} (U_i^{-1})_{\sigma\sigma'}\psi^{(i,\sigma')} \tag{A.3}$$

i.e.

$$\psi^i \to U_i^{-1}\psi^i; \qquad \psi \to U^{-1}\psi \tag{A.4}$$

where U_i is an arbitrary non-singular matrix of dimension α_i, and U is block diagonal, being the direct sum of the blocks U_i. This freedom can be used to simplify the forms of the β^μ without any loss of generality. The effect of the change of basis (A.4) on the β^μ is to transform the skeleton matrix C

$$C \to U^{-1}CU; \qquad \text{i.e. } C^{(\tau_i,\tau_j)} \to U_i^{-1}C^{(\tau_i,\tau_j)}U_j. \tag{A.5}$$

In the particular case where there are just three distinct IRs, the transformation is

$$\begin{aligned} &A \to U_1^{-1}AU_2, \qquad &&B \to U_1^{-1}BU_3 \\ &D \to U_2^{-1}DU_1, \qquad &&E \to U_3^{-1}EU_1. \end{aligned} \tag{A.6}$$

We show now that U_1, U_2, U_3 can be so chosen as to bring D and E to the forms (4.4a) and (4.4b).

Recall first that E is an $\alpha_3 \times \alpha_1$ matrix ($\alpha_3 \leqslant \alpha_1$) and that its rank is α_3. Therefore $(\alpha_1 - \alpha_3)$ of its columns can be made zero by forming suitable linear combinations of the α_1 columns, which is equivalent to post multiplication of E by a suitable non-singular matrix. This matrix, say U_1', can be so chosen that

$$EU_1' = (0 \quad E') \tag{A.7}$$

where E' consists of α_3 linearly independent columns and constitutes a non-singular matrix. Therefore, if we take $U_1 = U_1'$, $U_2 = I$ and $U_3 = E'$ in (A.6) we find that

$$\begin{aligned} &A \to U_1'^{-1}A, \qquad &&B \to U_1'^{-1}BE' \\ &D \to DU_1', \qquad &&E \to (0, I). \end{aligned} \tag{A.8}$$

For notational simplicity we shall now refer to these transformed matrices as A, B, D, E. They have of course the same dimensions and ranks as the original matrices. We now seek to simplify D by further transformations of the type (A.6), which are to be restricted so as not to disturb the form $(0\ I)$ to which E has been reduced. It is readily seen that the most general transformation matrices U_1, U_2, U_3 which honour this restriction have the forms

$$U_1'' = \begin{pmatrix} P & Q \\ 0 & R \end{pmatrix}, \qquad U_2'' = S, \qquad U_3'' = R^{-1} \tag{A.9}$$

where P, Q, R, S are arbitrary matrices, subject to P, R, S being non-singular. The partitioning of U_1'' is conformable to that of E, so that the dimensions of P and R are $(\alpha_1 - \alpha_3)$ and α_3 respectively. Under the above transformation, the matrix D (also partitioned like E) goes over into

$$U_2''^{-1}DU_1'' \equiv S^{-1}(D_1D_2)\begin{pmatrix} P & Q \\ 0 & R \end{pmatrix}$$

$$\equiv S^{-1}(D_1P, D_1Q+D_2R). \tag{A.10}$$

If D_1 had rank less than $(\alpha_1-\alpha_3)$, it would be possible to choose a P which makes some of the columns of D_1P null; since all the elements standing above D_1P in C are null and also those below (forming the null part of E), this would mean that some of the columns of C itself would become null. But this cannot be, if there are to be no barnacles. Therefore, the rank must be equal to $(\alpha_1-\alpha_3)$, the number of columns; these columns must be a linearly independent set. Since the rank of $D\equiv(D_1, D_2)$ is α_2, D_2 must contain a set of $\alpha_2-(\alpha_1-\alpha_3)$ linearly independent columns, which, together with the columns of D_1, make up the requisite α_2 linearly independent columns of D_1. These columns can be brought to the leading positions (i.e. positions immediately following D_1P), and the remaining $\alpha_3-[\alpha_2-(\alpha_1-\alpha_3)]\equiv\alpha_1-\alpha_2$ columns made null by choosing appropriately the post multiplying matrices Q and R (which effect column combinations) in D_1Q+D_2R. The result is that $(D_1P,\ D_1Q+D_2R)$ takes the form $(D', 0)$ where D' is a square non-singular matrix of dimension α_2. Choosing $S=D'$ one gets the final reduced form $(I, 0)$ for D:

$$D\to(I,0) \tag{A.11}$$

With these 'standard' forms for D and E, the matrix C takes the form

$$C=\begin{array}{c} \left(\begin{array}{ccc|cc|cc} 0 & 0 & 0 & A_1 & A_2 & B_1 & B_2 \\ 0 & 0 & 0 & A_3 & A_4 & B_3 & B_4 \\ 0 & 0 & 0 & A_5 & A_6 & B_5 & B_6 \\ \hline I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{c} \alpha_1' \\ \alpha_1'' \\ \alpha_1''' \\ \alpha_2' \\ \alpha_2'' \\ \alpha_3' \\ \alpha_3'' \end{array} \\ \begin{array}{ccccccc} \alpha_1' & \alpha_1'' & \alpha_1''' & \alpha_2' & \alpha_2'' & \alpha_3' & \alpha_3'' \end{array} \end{array} \tag{A.12}$$

$$\alpha_1'=\alpha_2'=\alpha_1-\alpha_3; \qquad \alpha_1''=\alpha_2''=\alpha_3'=\alpha_2+\alpha_3-\alpha_1; \qquad \alpha_1'''=\alpha_3''=\alpha_1-\alpha_2. \tag{A.13}$$

The dimensions of the subblocks in the partitioned form above are as indicated by the primed α's. For example A_3 is an $\alpha_1''\times\alpha_2'$ matrix.

The next question is how far A and B can be simplified (without loss of generality) while leaving the canonical forms of D and E intact. It is easy to verify that the general forms of transformation matrices U_1, U_2, U_3, which can be used for this purpose are

$$U_1=\begin{pmatrix} P & Q & 0 \\ 0 & R & 0 \\ 0 & S & T \end{pmatrix}, \qquad U_2=\begin{pmatrix} P & Q \\ 0 & R \end{pmatrix}, \qquad U_3=\begin{pmatrix} R & 0 \\ S & T \end{pmatrix}$$

with P, R, T non-singular. In the general case, the effect of these transformations on A and B is rather complicated, and there are several possible ways of proceeding. However, when specific types of theories are considered, such transformations can be used effectively to distinguish between inequivalent possibilities.

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